Model Questions

September 2010 **1^{st} Quiz #**

Test the following series for convergence

$$\sum_{n=2}^{\infty}\frac{Ln(n)}{n} \qquad \sum_{n=1}^{\infty}\frac{n^n}{3^{(1+2n)}} \qquad \sum_{n=1}^{\infty}\frac{1}{4^{2n+1}(n+1)}$$

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September 2010 **1^{st} Quiz @**

Test the following series for convergence

$$\sum_{n=1}^{\infty} n\,e^{n^2} \qquad \sum_{n=1}^{\infty}\left[\frac{2n^2+1}{n^2+1}\right]^n \qquad \sum_{n=1}^{\infty}\frac{n^2}{(2n-1)!}$$

October 2010 **2^{nd} Quiz**

1- Find Envelope of $f(x, y, t) = \frac{x}{t} + \frac{y}{1-t} = 1$, t is the parameter

2- Find the dimensions of the rectangular box with the largest volume if the total surface area is 64 cm^2.

October 2010 **3^{rd} Quiz**

1- Test the following series

$$\sum_{n=2}^{\infty}\frac{\tan^{-1}n}{1+n^2} \qquad \sum_{n=1}^{\infty}\frac{1}{n3^n} \qquad \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\left(1+\frac{1}{n}\right)^{n^2}$$

2- Classify the following series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^2}{n^2+3} \qquad \sum_{n=1}^{\infty}\frac{2n(-1)^{n-1}}{4n^2-3} \qquad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(n+1)}{5^n}$$

3- Find interval of convergence

$$\sum_{n=1}^{\infty}\frac{3^n x^{n-1}}{n!} \qquad \sum_{n=1}^{\infty}\frac{(-1)^n x^n (n!)^2}{3^n} \qquad \sum_{n=1}^{\infty}\frac{3^n}{(n^2+1)(x-2)^n}$$

October 2010 **4^{th} Quiz**

1- Find interval of convergence for the following series:

$$\frac{1}{1.6}x^2+\frac{2}{6(20)}x^4+\frac{6}{20(252)}x^6+\ldots.$$

2- Test the series $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^{3/4}}$

3- Expand using Taylor the function f(x,y) = (x +y)sinxy about (1,0)

October 2010 **5^{th} Quiz**

1- Solve the following differential equation

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2} \qquad \frac{dy}{dt} = 1 + t^2 + y^2 + t^2y^2 \qquad \frac{dy}{dx} = \frac{2x + 3y + 7}{4x + 6y + 28}$$

2- Find the first and second derivatives for the function

$$x\cos(xy) + e^{xy} = 0$$

3- Find all relative extrema and saddle points for

$$f(x, y) = 3x^2 + y^2 + 9x - 4y + 6$$

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December 2010 **12^{th} exam**

1- Determine if the following series is convergent or divergent: $\sum_{n=1}^{\infty}[\frac{5n-3n^3}{7n^3+2}]^n$, $\sum_{n=1}^{\infty}\frac{5^n+7^n}{3^n+2^n}$, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,2^n}$

2- Find the critical points of the function

$$f(x, y) = x^3 + x\,y^2 - 3x^2 - 4\,y^2 + 4$$

3- Solve the following differential equations:

a) $(y + \ln(x))dx + (x+y^2)\,dy = 0$ b) $y` = (y/2x) - (xy)^3$

c) $y`` + y = 1+ \tan x$ d) $y`` + 2y` + 2y = e^x \sin^2(2x)$

4- Solve the D.E. $y``- xy = x+4$ using series solution about $x = x_0$

5- Find grad (div(curl $\vec{u}$) given $\vec{u} = (xy + z\tan x)i + x^2ye^zj - (y\sin(xz))k$

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Correct the wrong statements giving the reason

a) If $\lim_{n\to\infty} U_n = 0$, then the Series $\sum_{n=1}^{\infty} U_n$ is convergent.

b) The Series $\sum_{n=1}^{\infty} P^n$ is convergent if $|P| < 1$.

c) The Series $\sum_{n=1}^{\infty} \frac{(-1)^n}{4^{2n+1}(n+1)}$ is conditionally convergent.

d) The differential equation $(y``)^2 + (y`)^5 + y = 0$ of order 2 and degree 5.

e) The envelope of straight line $x\cos\alpha + y\sin\alpha = P$ is a circle with center (α, α).

f) The minimal distance of the point (0,0,-1) from the plane given by z=2x–y+ 1 is (-2/3, 1/3, 4/3).

g) The integrating factor of the differential equation $xy\,dx + (x^2 + y)\,dy = 0$ is x.

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Model answer

a) **Wrong**, since the series $\sum_{n=1}^{\infty} U_n$ may be divergent if $\lim_{n\to\infty} U_n = 0$, e.g. $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and $\lim_{n\to\infty} \frac{1}{n} = 0$.

b) **Correct**, since $\sum_{n=1}^{\infty} P^n$ is convergent if 0 < P < 1 while $\sum_{n=1}^{\infty} P^n$ is absolutely convergent if -1< P < 0.

c) **Wrong**, $U_n = \frac{1}{4^{2n+1}(n+1)}$ and $U_{n+1} = \frac{1}{4^{2n+3}(n+2)}$, therefore $U_n > U_{n+1}$ and $\lim_{n\to\infty} \frac{1}{4^{2n+1}(n+1)} = 0$ and $U_n = \frac{1}{4^{2n+1}(n+1)}$ is convergent, thus $\sum_{n=1}^{\infty} \frac{(-1)^n}{4^{2n+1}(n+1)}$ is absolutely convergent.

d) **Wrong**, The D.E. $(y``)^2 + (y`)^5 + y = 0$ of order 2 and degree 2.

e) **Wrong**, since the envelope is $x^2 + y^2 = P^2$ which is circle with center (0,0).

f) **Wrong**, $f(x,y,z) = x^2 + y^2 (z+1)^2$, $\phi(x,y,z) = z-2x+y= 1$ and $f_x= \lambda \phi_x$, $f_y= \lambda \phi_y$ and $f_z= \lambda \phi_z$, therefore x = -2y and z = y-1, thus the nearest point is (-2/3, 1/3, -2/3) and the minimal distance is $\sqrt{\frac{2}{3}}$.

g) **Wrong**, The integrating factor of the differential equation $xy\,dx + (x^2 + y)\,dy = 0$ is y.

Faculty of Engineering (Shoubra) **Benha University** **Time allowed: 3 hours**	**1st year Production Mechanical Department**	**Engineering Mathematics and Physics Department** **16th of January 2011**

Question 1

a- Test the following series for convergence**:**

i) $\sum_{n=1}^{\infty} \frac{5^n + 7^n}{3^n + 2^n}$ ii) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n (2n!)^2}$ iii) $\sum_{n=1}^{\infty} \frac{e^n}{e^{2n} + 1}$

b- Find minimal distance of the point (0, 0, -1) from the plane given by $z = 2x - y + 1$

c- Solve the following differential equations:

i) $(y + \ln(x))dx + (x + y^2)\, dy = 0$, ii) $y`` + y = \sec(x)$,

iii) $y` = (y/2x) - (xy)^3$

Question 2

a- Given $\bar{R} = (x, y, z)$ so that $r = |\bar{R}| = \sqrt{x^2 + y^2 + z^2}$. Show that $\nabla (r^n) = n\, r^{n-2}\, \bar{R}$, for any integer n, then deduce grad (r), grad (r^2), grad (1/r).

b- Define: Sequence - Cauchy sequence – Order and degree of D.E. – Homogeneous function – Homogeneous D.E.

c- Verify Green's Theorem to evaluate $\int_C xy\, dx + x^2 y^3 dy$, where c is the triangle whose vertices (0,0), (1,0), (1,2) with positive orientation.

Question 3

a- Expand the function $f(x, y) = \tan^{-1}(\frac{x+y}{x-y})$ using Taylor series about (0,1)

b- Solve the D.E. $9x^2 y'' - (4+x) y = 0$ using series solution.

c- Solve the following differential equations:

i) $\frac{dy}{dx} = \frac{x+y+3}{x-y+1}$ ii) $y'' + 2y' + 2y = e^x \sin^2(2x)$

iii) $y'' + 5y' + 6y = 2-x+3x$

Question 4

a- Find envelope of the function $f(x, y, t) = \frac{x}{t} + \frac{y}{1-t} = 1$

b- Convert $y' + \phi(x)\, y = \psi(x)\, y^n$ into linear D.E.

c- Evaluate the following integrals

i) $\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx$, ii) $\int_{(0,0)}^{(1,-2)} (x+3y)dx + (3x-2y)dy$, the contour is $y^3 + 5x^3 + 3x = 0$

Question 5

a- Determine the critical points and locate any relative minima, maxima and saddle points of function f defined by

$$f(x , y) = - x^4 - y^4 + 4xy$$

b- Find the volume of the parallelepiped spanned by the vectors

$$u = (1,0,2) \qquad v = (0,2,3) \qquad w = (0,1,3)$$

If F = (x^2y, yz, x + z). Find (i) curl curl F; (ii) grad div F

c- Find interval of convergence for the following power series

$$\text{i)} \sum_{n=1}^{\infty} \frac{3^n}{(n^2+1)(x-2)^n} \qquad \text{ii)} \sum_{n=1}^{\infty} \frac{n\, e^{-n^2}}{x^n}$$

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Answer of Question 1

a-i) By ratio test, we get that $\lim_{n\to\infty} \left(\frac{5^{n+1}+7^{n+1}}{3^{n+1}+2^{n+1}}\right)\left(\frac{3^n+2^n}{5^n+7^n}\right)$ =

$\lim_{n\to\infty} \frac{7^{n+1}}{3^{n+1}}\left[\frac{(5/7)^{n+1}+1}{1+(2/3)^{n+1}}\right] \frac{3^n}{7^n}\left[\frac{3^n+2^n}{5^n+7^n}\right]$ = 7/3 > 1, therefore the series is divergent. (3 marks)

ii) Let $U_n = \frac{1}{3^n(2n!)^2}$, $\lim_{n\to\infty} \frac{1}{3^n(2n!)^2} = 0$, $U_{n+1} = \frac{1}{3^{n+1}((2n+2)!)^2}$, hence $U_n > U_{n+1}$. By using ratio test, we will get that $\sum_{n=1}^{\infty} \frac{1}{3^n(2n!)^2}$ is convergent, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n(2n!)^2}$ is called absolutely convergent. (2 marks)

iii) Since $\int_1^\infty \frac{e^n}{e^{2n}+1}dn = (\tan^{-1}e^n)_1^\infty = \tan^{-1}\infty - \tan^{-1}e = -\tan^{-1}e$, thus $\sum_{n=1}^\infty \frac{e^n}{e^{2n}+1}$ is convergent. (2 marks)

b) Let the point on the plane is (x, y, z) and f(x, y, z) = $x^2 + y^2 + (z+1)^2$ is the square of the distance between (0, 0, -1) and (x, y, z), also ϕ(x, y, z) = z – 2x + y =1. By applying conditional extrema, we get $f_x=\lambda\phi_x$, $f_y=\lambda\phi_y$, $f_z=\lambda\phi_z$, thus 2x = λ (– 2), 2y = λ (1) and 2(z+1) = λ(1), therefore x =-2y and z = y-1. Substitute in ϕ (x, y, z) = z – 2x + y =1 so that (-2/3, 1/3, -2/3) is the point on plane and the minimal distance from point (0, 0, -1) = $\sqrt{2/3}$. (7 marks)

c-i) (y + ln(x))dx +(x+y^2) dy = 0 is exact D.E. since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ = 1, therefore $\frac{\partial f}{\partial x} = M(x,y) =$ (y + ln(x))$\Rightarrow$f(x,y) = xy + xLn x – x + $\phi(y)$, thus $\frac{\partial f}{\partial y} = x + \phi'(y) =$ x+y^2, hence $\phi(y)$ = y`` + 2y` + 2y = e^x $\sin^2(2x)y^3/3$, therefore f(x,y) = xy + xLn x – x + $y^3/3$+ c (3 marks)

ii) y`` + y = sec(x) has homogeneous and particular solution so that the characteristic equation is $m^2+1=0 \Rightarrow$ m =-i, i, thus y_H= (c_1 cosx + c_2 sinx) and so the particular solution is: $y_P = u_1(x)\cos x + u_2(x)\sin x$, and $y_1(x)$ = cosx, $y_2(x)$ = sinx where $u_1(x) = -\int \frac{y_2 g(x)}{W(y_1, y_2)}dx$,

$u_2(x) = \int \frac{y_1 g(x)}{W(y_1, y_2)} dx$, where $W(y_1, y_2) = y_2' y_1 - y_2 y_1' = 1$, g(x) = secx,

thus $u_1(x) = -\int \frac{\sin x \sec x}{1} dx = \text{Ln} \cos x$ & $u_2(x) = \int \frac{\cos x \sec x}{1} dx = x$

(2 marks)

iii) $y' = (y/2x) - (xy)^3$ is Bernoulli D.E., thus $y^{-3}y' - y^{-2}/2x = -x^3$. Put $z = y^{-2} \Rightarrow z' = -2y^{-3}y' \Rightarrow z' + z/x = 2x^3$ which is linear D.E. whose solution is $zx = -2x^5/5 + c$, so $xy^{-2} = -2x^5/5 + c$ is the solution of D.E. (2 marks)

Answer of Question 2

a- $\nabla(r^n) = \nabla(x^2 + y^2 + z^2)^{n/2}$

$= \frac{n}{2}(x^2 + y^2 + z^2)^{(n/2)-1}(2x\vec{i} + 2y\vec{j} + 2z\vec{k}) = n\, r^{n-2}\overline{R}$

At n = 1 $\Rightarrow$ grad (r) = $\frac{1}{r}\overline{R}$, and at n = 2 $\Rightarrow$ grad $(r^2) = 2\overline{R}$, finally at n = -1 $\Rightarrow$ grad (1/r) = $-\frac{1}{r^3}\overline{R}$. (7 marks)

b- Sequence**:** group of elements related by general term whose domain is set of positive integers.

Cauchy Sequence: Every convergent sequence is called Cauchy sequence.

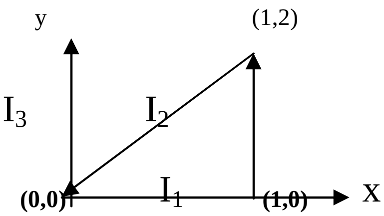
Order of D.E. : is the highest derivative of D.E.

Degree of D.E. : is the power of the highest derivative of D.E.

Homogeneous function: f(x,y) is homogeneous of order n if $f(tx, ty) = t^n f(x,y)$.

Homogeneous D.E. : is the D.E. in which the particular solution equal zero. (7 marks)

c)



If we express the problem in line integral, we have to divide the path into three paths $I_1 : y = 0$, $I_2 : x = 1$, $I_3 : 2x = y$ so that $dy = 0$, $dx = 0$, $2dx = dy$ respectively.

For path I_1: $\int_{I_1} xy\,dx + x^2y^3dy = \int_0^1 xy\,dx + x^2y^3dy = 0$

For path I_2: $\int_{I_2} xy\,dx + x^2y^3dy = \int_0^2 y^3dy = 4$

For path I_3: $\int_{I_3} xy\,dx + x^2y^3dy = \int_1^0 [x(2x) + x^2(2x)^3(2)]dx = -\dfrac{10}{3}$

Therefore $\int_c xy\,dx + x^2y^3dy = I_1 + I_2 + I_3 = \dfrac{2}{3}$.

By using Green theorem

$$\int_c xy\,dx + x^2y^3dy = \iint_D (2xy^3 - x)\,dxdy = \int_{x=0}^{1}\int_{y=0}^{y=2x} (2xy^3 - x)\,dydx$$

$$= \int_{x=0}^{1} (8x^5 - 2x^2)dx = \frac{2}{3}$$ (7 marks)

Answer of Question 3

a- Since $f(x, y) = \tan^{-1}(\frac{x+y}{x-y})$, then $f_x = \frac{-y}{x^2+y^2}$, $f_y = \frac{x}{x^2+y^2}$, f_{xx} $= \frac{2xy}{(x^2+y^2)^2}$, $f_{xx} = \frac{2xy}{(x^2+y^2)^2}$, $f_{yy} = \frac{-2xy}{(x^2+y^2)^2}$, and $f_{xy} = \frac{y^2 - x^2}{(x^2+y^2)^2}$.

At (0,1) , $f(0,1) = -\frac{\pi}{4}$, $f_x = -1$, $f_y = 0$, $f_{xx} = f_{yy} = \frac{2xy}{(x^2+y^2)^2} = 0$, $f_{xy} = 1$,

then by substituting in Taylor formula, $f(x, y) = -\frac{\pi}{4} - x + x(y-1)$.

b- We note that the series solution at $x = x_0$, $x_0 \neq 0$ still in regular case, so that the solution can be expressed as below, but $p(x) = 0$ & $q(x) = \frac{-(4+x)}{9x^2}$ are not analytic at $x = 0$, therefore $x = 0$ is called regular singular since $xp(x)$ & $x^2q(x)$ are analytic at $x = 0$, then series solution about $x = 0$ can be expressed in the form:

$$y(x)=\sum_{n=0}^{\infty} a_n x^{n+c},\quad y'(x)=\sum_{n=0}^{\infty}(n+c)\,a_n x^{n+c-1},$$

$$y''(x)=\sum_{n=0}^{\infty}(n+c)(n+c-1)\,a_n x^{n+c-2},.....,$$

Substitute in the above D.E. , so we get

$$9x^2\sum_{n=0}^{\infty}(n+c)(n+c-1)\,a_n x^{n+c-2}-(4+x)\sum_{n=0}^{\infty}a_n x^{n+c}=0 \quad \Rightarrow$$

$$9\sum_{n=0}^{\infty}(n+c)(n+c-1)\,a_n x^{n+c}-4\sum_{n=0}^{\infty}a_n x^{n+c}-\sum_{n=0}^{\infty}a_n x^{n+c+1}=0$$

Put n = m-1 for the 3rd term, n = m for 1st & 2nd terms, we get

$$[9c(c-1)-4]\,a_0\,x^c+\sum_{m=1}^{\infty}([9(m+c)(m+c-1)-4]a_m-a_{m-1})x^{m+c}=0$$

By comparing of coefficients of x^c, therefore $[9c(c-1)-4]\,a_0=0$, $a_0\neq 0$, then $9c(c-1)-4=0$, from which $c_1=-1/3$, $c_2=4/3$, therefore $c_1-c_2\neq$ integer (case 1). By comparing of coefficients of x^{m+c}, therefore $a_m=\dfrac{a_{m-1}}{9(m+c)(m+c-1)-4}$.

At $c=-1/3$, $a_m=\dfrac{a_{m-1}}{9m^2-15m}$, m = 1, 2, ...

$a_1 = \frac{-a_0}{6}$, $a_2 = \frac{a_1}{6} = \frac{-a_0}{36}$, $a_3 = \frac{a_2}{36} = -\frac{a_0}{(36)^2}$, therefore

$$U(x) = x^{-1/3} a_0 \left[1 - \frac{1}{6} x - \frac{1}{36} x^2 - \frac{1}{(36)^2} x^3 + \ldots\right]$$

At c = 4/3, $a_m = \frac{a_{m-1}}{9m^2 + 15m}$, $a_1 = \frac{a_0}{24}$, $a_2 = \frac{a_1}{66} = \frac{a_0}{1584}$, $a_3 = \frac{a_2}{126} = \frac{a_0}{199584}$, therefore $V(x) = x^{4/3} a_0 \left[1 + \frac{1}{24} x + \frac{1}{1584} x^2 + \frac{1}{199584} x^3 + \ldots\right]$, thus $Y(x) = Aa_0 x^{-1/3} \left[1 - \frac{1}{6} x - \frac{1}{36} x^2 - \frac{1}{(36)^2} x^3 + \ldots\right] + B x^{4/3} a_0 \left[1 + \frac{1}{24} x + \frac{1}{1584} x^2 + \frac{1}{199584} x^3 + \ldots\right]$ (7 marks)

c- i) Since $\frac{dy}{dx} = \frac{x+y+3}{x-y+1}$ is non homogeneous equation. To solve this differential equation, we have to follow these steps

(1) We have to get the point of intersection between x+y+3=0, x-y+1=0 which is (-2,-1),

(2) Put x = X-2, y = Y-1, dx = dX, dy = dY in the above differential equation, then $\frac{dY}{dX} = \frac{X+Y}{X-Y}$, so it is a homogeneous equation,

(3) Put Y = vX & dY = vdX + Xdv, thus $\frac{vdX+Xdv}{dX} = \frac{X+vX}{X-vX} = \frac{1+v}{1-v}$

(4) Integrate $\frac{dX}{X}=\frac{(1-v)dv}{1+v^2}$, then put X=x+2, $v=\frac{Y}{X}=\frac{y+1}{x+2}$ so that the solution of the differential equation is $\text{Ln}(x+2) = \tan^{-1}(\frac{y+1}{x+2}) - \frac{1}{2}\ln(\frac{(y+1)^2+(x+2)^2}{(x+2)^2}) + C$ (3 marks)

ii) $y'' + 2y' + 2y = e^x \sin^2(2x)$ has homogeneous and particular solution so that the characteristic equation is $m^2 + 2m + 2 = 0 \Rightarrow m = -1 \pm i$, thus $y_H = e^{-x}(c_1\cos x + c_2 \sin x)$ and so the particular solution is $y_P = \frac{1}{D^2+2D+2} e^x \sin^2 x = \frac{1}{D^2+2D+2}(\frac{e^x}{2})(1-\cos 2x)$

$= \frac{e^x}{2}[\frac{1}{5}-\frac{(8\sin 2x+\cos 2x)}{65}]$ (2 marks)

iii) $y'' + 5y' + 6y = 2 - x + 3x^2$ has homogeneous and particular solution so that the characteristic equation is $m^2 + 5m + 6 = 0 \Rightarrow m = -2, -3$, thus $y_H = c_1 e^{-2x} + c_2 e^{-3x}$)and so the particular solution is $y_P =$

$\frac{1}{D^2+5D+6}(2-x+3x^2) = \frac{1}{6}(1+\frac{D^2+5D}{6})^{-1}(2-x+3x^2) = \frac{1}{6}(6\text{-}6x+3x^2)$

(2 marks)

Answer of Question 4

a- Differentiate w.r.t. t such that $-x/t^2 - y(1-t)^{-2}(-1) = 0$, thus $t = \frac{1}{1 \pm \sqrt{\frac{y}{x}}}$, & $1-t = \frac{\pm\sqrt{\frac{y}{x}}}{1 \pm \sqrt{\frac{y}{x}}}$, thus the envelope is $(1 \pm \sqrt{\frac{y}{x}})\,(x \pm \sqrt{xy}) = 1$

(7 marks)

b- A differential equation of Bernoulli type is written as $y` + \phi(x)\,y = \psi(x)\,y^n$

This type of equation is solved via a substitution. Indeed, let $z = y^{1-n}$, then easy calculations give $z` = (1-n)\,y^{-n}\,y`$ which implies $\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x)$. (7 marks)

c- i) Put $x = r\cos\theta$, $y = r\sin\theta$, therefore $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_{0}^{2}\int_{0}^{\pi} r^2\,dr\,d\theta = \frac{8}{3}\pi$ (3 marks)

ii) Since $P_y = 3 = P_x$, therefore the integral independent on the path, thus we can take the path as the line joining between the two end points such that $y = -2x$, hence $dy = -2dx$, therefore $\int_{(0,0)}^{(1,-2)} (x+3y)dx + (3x-2y)dy = \int_{0}^{1} -19x\,dx = -19/2$

Answer of Question 5

a- f_x=-4 x^3 + 4y = 0, f_y=-4 y^3 + 4x = 0, therefore y = x^3, substitute in one of the two equations, hence (0,0), (1,1), (-1,-1) are the critical points and f_{xx} = -12 x^2, f_{yy}= -12 y^2, f_{xy} = 4. At (0,0), Δ = -16 < 0 (saddle point) and at $(\pm 1, \pm 1)$, Δ = 128 > 0, f_{xx}, f_{yy} < 0 (maximum point) (7 marks)

b- Volume = $\vec{u} \bullet (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 1 & 3 \end{vmatrix} = 3.$

$$\text{Curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & yz & x+z \end{vmatrix} = -y\,\vec{i} - \vec{j} - x^2\,\vec{k}$$

$$\text{Curl Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & -1 & -x^2 \end{vmatrix} = 2x\,\vec{j} + \vec{k}$$

Div $\vec{F}$ = 2xy+z+1, grad div $\vec{F}$ = $2y\,\vec{i} + 2x\,\vec{j} + \vec{k}$ (7 marks) c- i) Since $U_n = \frac{3^n}{(n^2+1)(x-2)^n}$ & $U_{n+1} = \frac{3^{n+1}}{((n+1)^2+1)(x-2)^{n+1}}$, hence

$\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{3^{n+1}(n^2+1)(x-2)^n}{3^n((n+1)^2+1)(x-2)^{n+1}}\right| = \left|\frac{3(n^2+1)}{((n+1)^2+1)(x-2)}\right|$, thus $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right|$

$= \lim_{n\to\infty}\left|\frac{3(n^2+1)}{((n+1)^2+1)(x-2)}\right| = \lim_{n\to\infty}\left|\frac{3}{(x-2)}\right| < 1$ to be convergent, hence

$|x-2| > 3$, thus x > 5 or x < -1 is the interval of convergence. (7marks)

ii) Since $U_n = \frac{n\,e^{-n^2}}{x^n}$, and $U_{n+1} = \frac{(n+1)\,e^{-(n+1)^2}}{x^{n+1}}$, hence the ratio

$\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{x^n(n+1)\,e^{-(n+1)^2}}{x^{n+1}\quad n\,e^{-n^2}}\right| = \left|\frac{(n+1)}{ne^{(2n+1)}x}\right|$, therefore $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| =$

$\lim_{n\to\infty}\left|\frac{(n+1)}{ne^{(2n+1)}x}\right| = 0 < 1$, so $\sum_{n=1}^{\infty}\frac{n\,e^{-n^2}}{x^n}$ is convergent for all x. (3 marks)

Faculty of Engineering (Shoubra) Engineering Mathematics & Physics Department Mid term exam 2011-2012		Benha University Mechanical Department - 1st year Production Time allowed: 1 hour
Name in Arabic:	Section:	B.N.

I) Solve 3 only of the following D.E.

i) $y' = \frac{2 + ye^{xy}}{2y - xe^{xy}}$, ii) $1 - \frac{x}{x^2 + y^2} - \frac{yy'}{x^2 + y^2} = 0$, iii) $y^2 - xy + x^2 y` = 0$, iv) $y` = \frac{2x - 3y + 9}{6y - 4x + 1}$, v) $y``` - 4y`` + 5y` - 2y = 0$

II) Solve 3 of the following questions

i) Test for convergence:

$$\sum_{n=1}^{\infty} \frac{3^n}{((2n+1)!)^n}, \quad \sum_{n=1}^{\infty} \frac{4^n + 9^n}{5^n + 11^n}, \quad \sum_{n=1}^{\infty} \frac{2n(-1)^{n-1}}{4n^2 - 3}$$

ii) Expand the function $f(x,y) = e^{xy}\cos(x+y)$ about $(\pi/4, \pi/4)$ using Taylor expansion.

iii) Find envelope of $y = \alpha x + 1/\alpha$

iv) Determine the critical points and locate any relative minima, maxima and saddle points of function $f(x, y) = -x^4 - y^4 + 4xy$.

Dr. Eng. Khaled El Naggar

Quiz1 1^{st} year Production **October 2011**

If $\lim_{n\to\infty} U_n = 0$, then the Series $\sum_{n=1}^{\infty} U_n$ is convergent

(Correct if it is wrong)

Ans. Wrong because $\lim_{n\to\infty} U_n = 0$ is necessary but not sufficient condition for the series to be convergent.

$\sum_{n=1}^{\infty} \frac{n}{4^{2n+1}(n+1)}$ (Test for convergence)

Ans. $U_{n+1} = \frac{n+1}{4^{2n+3}(n+2)}$, so the ratio $\frac{U_{n+1}}{U_n} = \frac{(n+1)(n+1)4^{2n+1}}{4^{2n+3}(n+2)n}$

Therefore $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{1}{16} < 1$, thus $\sum_{n=1}^{\infty} \frac{n}{4^{2n+1}(n+1)}$ is convergent.

The series $\sum_{n=1}^{\infty} P^n$ is convergent if |P| > 1. (Correct if it is wrong)

Ans. Wrong because the series $\sum_{n=1}^{\infty} P^n$ is divergent if |P| > 1

$$\sum_{n=1}^{\infty}[\frac{2n^2+1}{n^2+1}]^n \qquad \text{(Test for convergence)}$$

Ans. $\lim_{n\to\infty}\sqrt[n]{(\frac{2n^2+1}{n^2+1})^n} = \lim_{n\to\infty}(\frac{2n^2+1}{n^2+1}) = 2 > 1$, thus $\sum_{n=1}^{\infty}[\frac{2n^2+1}{n^2+1}]^n$ is divergent

If the sequence $\{U_n\}_{n=1}^{\infty}$ is convergent, then $\lim_{n\to\infty} U_n = 0$.

(Correct if it is wrong)

Ans. Wrong because if $\lim_{n\to\infty} U_n = L \in \Re$, then the sequence $\{U_n\}_{n=1}^{\infty}$ is convergent

$$\sum_{n=1}^{\infty} n\, e^{n^2} \qquad \text{(Test for convergence)}$$

Ans. $\int_1^{\infty} n\, e^{n^2}\, dn = \frac{1}{2}\int_1^{\infty} 2n\, e^{n^2}\, dn = \frac{1}{2}(e^{n^2})_1^{\infty} = \infty$, thus $\sum_{n=1}^{\infty} n\, e^{n^2}$ is divergent

The series $\sum_{n=1}^{\infty} P^{-n}$ is convergent if | P| < 1. (Correct if it is wrong)

Ans. Wrong because the series $\sum_{n=1}^{\infty} P^{-n}$ is divergent if | P| < 1

$$\sum_{n=1}^{\infty} \frac{4n^2 + 3n}{\sqrt[7]{n^5 + n^3}}$$ (Test for convergence)

Ans. The series $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} n^{9/7}$ is divergent and $\lim_{n \to \infty} \frac{U_n}{V_n} = 4$, therefore $\sum_{n=1}^{\infty} \frac{4n^2 + 3n}{\sqrt[7]{n^5 + n^3}}$ divergent

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If $\sum_{n=1}^{\infty} U_n$ is divergent, then $\lim_{n \to \infty} U_n \neq 0$. (Correct if it is wrong)

Ans. Wrong as if $\lim_{n \to \infty} U_n \neq 0$, then $\sum_{n=1}^{\infty} U_n$ is divergent, and if the series $\sum_{n=1}^{\infty} U_n$ may be divergent, then $\lim_{n \to \infty} U_n = 0$ or $\lim_{n \to \infty} U_n \neq 0$.

$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ (Find interval of convergence)

Ans. $U_n = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ & $U_{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n+2}}{2^{2n+2}((n+1)!)^2}$, hence the ratio $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{(-1)^{n+1} x^{2n+2}}{2^{2n+2}((n+1)!)^2} \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| = \left|-\frac{x^2}{4(n+1)^2}\right|$, thus $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|-\frac{x^2}{4(n+1)^2}\right| = 0$, i.e. the series is convergent for all x.

The series $\sum_{n=1}^{\infty} n^{-p}$ is convergent if P < 1. (Correct if it is wrong)

Ans. Wrong because The series $\sum_{n=1}^{\infty} n^{-p}$ is convergent if P > 1

$\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n(2n!)^2}$ (Classify this alternating series)

Ans. Let $U_n = \frac{1}{3^n(2n!)^2}$, $\lim_{n\to\infty}\frac{1}{3^n(2n!)^2}=0$, and $U_{n+1} = \frac{1}{3^{n+1}((2n+2)!)^2}$, hence $U_n > U_{n+1}$, but $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\frac{3^n(2n!)^2}{3^{n+1}((2n+2)!)^2}=0$, therefore we will get that $\sum_{n=1}^{\infty}\frac{1}{3^n(2n!)^2}$ is convergent, so $\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n(2n!)^2}$ is called absolutely convergent.

If $\lim_{n\to\infty} U_n \neq 0$, then series $\sum_{n=1}^{\infty} U_n$ is convergent. (Correct if it is wrong)

Ans. Wrong because if $\lim_{n\to\infty} U_n \neq 0$, then series $\sum_{n=1}^{\infty} U_n$ is divergent

$$\sum_{n=1}^{\infty} \frac{4^n + 9^n}{5^n + 11^n} \qquad \text{(Test for convergence)}$$

Ans. $U_{n+1} = \dfrac{4^{n+1} + 9^{n+1}}{5^{n+1} + 11^{n+1}}$, then $\dfrac{U_{n+1}}{U_n} = \dfrac{(4^{n+1} + 9^{n+1})(5^n + 11^n)}{(4^n + 9^n)(5^{n+1} + 11^{n+1})}$

$$= \frac{9^{n+1}((4/9)^{n+1} + 1)11^n((5/11)^n + 1)}{9^n((4/9)^n + 1)11^{n+1}((5/11)^{n+1} + 1)}, \text{so} \lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{9}{11} < 1,$$

therefore $\sum_{n=1}^{\infty} \dfrac{4^n + 9^n}{5^n + 11^n}$ is convergent.

Faculty of Engineering (Shoubra) Engineering Mathematics & Physics Department Mid term exam 2011-2012		Benha University Mechanical Department - 1st year Production Time allowed: 1 hour
Name in Arabic:		**Section:** **B.N.**

I) Solve the following D.E.

$$y'' + 3y' + 2y = \frac{1}{(1+e^x)^2}, \qquad y'' + 3y' - 4y = x\cosh 3x,$$

$$y''' - 4y'' + y' + 6y = 4\sin 2x$$

II) Find the first four terms in each portion of the series solution around x = 0 for the following differential equation $y'' + xy' = x^2 + 2$

Dr. Khaled El Naggar

Benha University	**Final Term Exam**
Faculty of Engineering- Shoubra	**Date:14th of January 2012**
Mechanical Engineering	**Mathematics 1**
1st Year Mechanical (production)	**Duration : 3 hours**

1-a) Test the following series for convergence:

$$\sum_{n=1}^{\infty}\left[\frac{5n^2-3n^3}{7n^3+2}\right]^{5n},\quad \sum_{n=1}^{\infty}\frac{n}{4^{2n+1}(n+1)},\quad \sum_{n=1}^{\infty}\frac{e^n}{e^{2n}+1},\quad \sum_{n=1}^{\infty}\frac{n^2}{\sqrt[5]{n^2+1}}$$

(12marks)

1-b) Solve the following differential equations:

$x\sec^2 y\, dx = e^{-x}dy$, $y` + y\tan x = y\cos^3 x$, $\dfrac{dy}{dx}=\dfrac{x+y+3}{x-y+1}$,

$x(6xy + 5)\, dx +(2x^3 + 3y)dy=0$

(12marks)

2-a) Expand the function $f(x, y) = \ln\left(\dfrac{x+y}{x-y}\right)$ using Taylor Maclaurin and Taylor series about (0,1)

(8 marks)

2-b) Find the interval of convergence for the following series:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n^2+3}\qquad \sum_{n=1}^{\infty}\frac{3^n}{(n^2+1)(x-2)^n}\qquad \sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^{2n}}{2n!}$$

(9 marks)

2-c) Find Envelope of $f(x, y, \alpha) = x\cos\alpha + y\sin\alpha = P$, α is the parameter

(6 marks)

2-d) Find the dimensions of the rectangular box with the largest volume with faces parallel to the coordinate planes that can be inscribed in the ellipsoid $16x^2 + 4y^2 + 9z^2 = 144$.

(6 marks)

3-a) Find the first four terms in each portion of the series solution for the following differential equation $y`` - xy = x+4$

(10 marks)

3-b) Find all relative extrema and saddle points for $f(x, y) = -x^2 - 4x - y^2 + 2y - 1$

(6 marks)

3-c) Solve the following differential equations:

i) $xydx + (x^2 + y)dy = 0$, ii) $y` - xy = e^{x^2} \cos x$,

iii) $(xy - x^2)\, dy - y^2 dx = 0$, iv) $(D^3 - 4D^2 + 5D - 2)y = 0$

(12marks)

4-a) Evaluate the following integrals

i) $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ ii) $\int_c \sin(\pi y)\, dx + yx^2 dy$, c is line segment from (0,2) to (1,4)

(12 marks)

4-b) Solve the following differential equations:

$y'' - 3y' + 2y = \frac{2}{1+e^{-x}}$, $y`` - 2y` + y = (x^2 - 1)\, e^{2x} + (3x+4)\, e^x$,

$y'' + 5y' + 4y = e^{5x} \cos 2x$

(12 marks)

BOARD OF EXAMINERS: *Dr.Eng. Khaled El Naggar*

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Model Answer

1a-i) $\lim_{n\to\infty} \sqrt[n]{[\frac{5n^2 - 3n^3}{7n^3+2}]^{5n}} = \lim_{n\to\infty} [\frac{5n^2 - 3n^3}{7n^3+2}]^5 = [-\frac{3}{7}]^5 < 1$, thus

$\sum_{n=1}^{\infty} [\frac{5n^2 - 3n^3}{7n^3+2}]^{5n}$ is convergent

ii) By ratio test, we get that $\lim_{n\to\infty} (\frac{n+1}{4^{2n+3}(n+2)})(\frac{4^{2n+1}(n+1)}{n})$

= 1/16 < 1, therefore the series is divergent.

iii) Since $\int_{1}^{\infty}\frac{e^{n}}{e^{2n}+1}dn=(\tan^{-1}e^{n})_{1}^{\infty}=\tan^{-1}\infty-\tan^{-1}e=-\tan^{-1}e$,

therefore $\sum_{n=1}^{\infty}\frac{e^{n}}{e^{2n}+1}$ is convergent.

iv) The series $\sum_{n=1}^{\infty}V_n=\sum_{n=1}^{\infty}n^{8/5}$ is divergent, thus $\sum_{n=1}^{\infty}\frac{n^2}{\sqrt[5]{n^2+1}}$ is convergent as $n^{8/5}>\frac{n^2}{\sqrt[5]{n^2+1}}$.

1b-i) By separation method, we get xe^{x} dx = $\cos^{2}y$dy, therefore $xe^{x}-e^{x}=[y+(\sin 2y)/2]/2$

ii) By separation method, we get dy/y + (tanx - $\cos^{3}$x)dx = 0, so dy/y + (tanx – (1-$\sin^{2}$x)cosx) dx = 0

Therefore the solution is lny + [-lncosx - sinx+ ($\sin^{3}$x)/3] = c

iii) Since $\frac{dy}{dx}=\frac{x+y+3}{x-y+1}$ is non homogeneous equation. To solve this differential equation, we have to follow these steps

(1) We have to get the point of intersection between x+y+3=0, x-y+1=0 which is (-2,-1),

(2) Put x=X-2, y=Y-1, dx=dX,dy=dY in the above differential equation, then $\frac{dY}{dX}=\frac{X+Y}{X-Y}$, so it is a homogeneous equation,

(3) Put Y=vX $\Rightarrow$ dY = vdX + Xdv, thus $\frac{vdX+Xdv}{dX}=\frac{X+vX}{X-vX}$ $=\frac{1+v}{1-v}$

(4) Integrate $\frac{dX}{X}=\frac{(1-v)dv}{1+v^2}$, then put X=x+2, $v=\frac{Y}{X}=\frac{y+1}{x+2}$ so that the solution of the D.E. is Ln(x+2) = $\tan^{-1}(\frac{y+1}{x+2})$ - $\frac{1}{2}\ln(\frac{(y+1)^2+(x+2)^2}{(x+2)^2})+C$

iv) $x(6xy + 5)\,dx + (2x^3 + 3y)dy=0$ is exact D.E. since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}= 6x^2$, thus $\frac{\partial f}{\partial x}=M(x,y)= x(6xy + 5)$, therefore $f(x,y) = 2x^3y + 5x^2/2 + \phi(y)$, thus $\frac{\partial f}{\partial y}=2x^3+\phi'(y) = 2x^3+ 3y$, hence $\phi(y) = 3y^2/2$, therefore solution is $f(x,y) = 2x^3y + 5x^2/2 +3y^2/2 + c$

2a) We have to get f_x, f_y , f_{xx} , f_{xy} , f_{yy} such that $f_x = \frac{1}{x+y} - \frac{1}{x-y}$, $f_y = \frac{1}{x+y} + \frac{1}{x-y}$, $f_{xx} = \frac{-1}{(x+y)^2} + \frac{1}{(x-y)^2}$ $f_{yy} = \frac{-1}{(x+y)^2} + \frac{1}{(x-y)^2}$, $f_{xy} = \frac{-1}{(x+y)^2} - \frac{1}{(x-y)^2}$

Therefore: at (0, 1), f (0, 1) = 0, f_x = 2, f_y = 0, f_{xx} =0, f_{yy} = 0, f_{xy} = -2, therefore f(x,y)= f(0, 0)+ $\frac{1}{1!}$ ($f_x(0,0)$ (x-0) + $f_y(0,0)$(y-0)) + $\frac{1}{2!}$ ($f_{xx}(0,1)(x-0)^2$ + 2(x-0) (y-1) $f_{xy}(0, 1)$+ $f_{yy}(0, 1)$ $(y-1)^2$) , therefore f(x,y) = 2x + x (y-1)

2b-i) Since $U_n = \frac{(-1)^{n-1} x^{2n}}{n^2+3}$, and $U_{n+1} = \frac{(-1)^n x^{2n+2}}{(n+1)^2+3}$, hence $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{(n^2+3)x^{2n+2}}{[(n+1)^2+3]x^{2n}}\right|$, thus $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{(n^2+3)x^2}{[(n+1)^2+3]}\right|$ = $\lim_{n\to\infty} x^2 < 1$ to be convergent, so $-1 < x < 1$ is the interval of convergence.

ii) Since $U_n = \frac{3^n}{(n^2+1)(x-2)^n}$, & $U_{n+1} = \frac{3^{n+1}}{((n+1)^2+1)(x-2)^{n+1}}$,

hence $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{3^{n+1}(n^2+1)(x-2)^n}{3^n((n+1)^2+1)(x-2)^{n+1}}\right| = \left|\frac{3(n^2+1)}{((n+1)^2+1)(x-2)}\right|$, Therefore $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{3(n^2+1)}{((n+1)^2+1)(x-2)}\right| = \lim_{n\to\infty}\left|\frac{3}{(x-2)}\right| < 1$ to be convergent, hence $|x-2| > 3$, thus x > 5 or x < -1 is the interval of convergence.

iii) Since $U_n = \frac{(-1)^n (x-1)^{2n}}{2n!}$, and $U_{n+1} = \frac{(-1)^{n+1}(x-1)^{2n+2}}{(2n+2)!}$,

hence $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{(-1)^{n+1}(x-1)^{2n+2}\ 2n!}{(-1)^n (x-1)^{2n}(2n+2)!}\right| = \left|\frac{(x-1)^2}{(2n+2)(2n+1)}\right|$, thus

$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^2}{(2n+2)(2n+1)}\right| = 0$ hence series is convergent for all x.

2-c) $\frac{\partial}{\partial\alpha}(x\cos\alpha + y\sin\alpha = P)$, therefore $-x\sin\alpha + y\cos\alpha = 0$, thus $\tan\alpha = y/x$, so $\cos\alpha = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\alpha = \frac{y}{\sqrt{x^2+y^2}}$, hence envelope is $x^2+y^2 = P^2$.

2-d) $f(x,y,z) = xyz$, $\phi(x,y,z) = 16x^2 + 4y^2 + 9z^2 = 144$ and $f_x = \lambda\phi_x$, $f_y = \lambda\phi_y$ and $f_z = \lambda\phi_z$, therefore $yz = \lambda(32x)$, xz = $\lambda(8y)$ & xy = $\lambda(18z)$, thus y = 2x , z = 4/3 x, thus $x = \sqrt{3}$, $y = 2\sqrt{3}$, $z = 4/3\sqrt{3}$, so the largest volume = $8\sqrt{3}$.

3-a) Consider; $y(x) = \sum_{n=0}^{\infty} a_n (x)^n$, $y'(x) = \sum_{n=1}^{\infty} n\, a_n (x)^{n-1}$, $y''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n (x)^{n-2}$. Substitute in the above D.E., we get $\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n$ = x+ 4, thus $\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1}$ = x+ 4. Put n = m+2 for 1st term and n = m-1 for 2nd term, thus $\sum_{m=0}^{\infty} (m+2)(m+1)\, a_{m+2} x^m - \sum_{m=1}^{\infty} a_{m-1} x^m$ = x+4, therefore $2a_2 + \sum_{m=1}^{\infty} [(m+2)(m+1)\, a_{m+2} - a_{m-1}] x^m$ = x+4, so a_2 = 2 & a_3 =[1+ a_0]/6, $(m+2)(m+1)\, a_{m+2} - a_{m-1} = 0$, m = 2, 3, …

3b) Find the first partial derivatives f_x and f_y such that $f_x(x,y) = -2x - 4$, $f_y(x,y) = -2y + 2$

Determine the critical points by solving the equations $f_x(x,y) = 0$ and $f_y(x,y) = 0$ simultaneously, hence $-2x - 4 = 0$,

$-2y+2 = 0$, therefore the critical point is (-2,1), then determine the second order partial derivatives such that: $f_{xx}(x,y) = -2$,

$f_{yy}(x,y) = -2$, $f_{xy}(x,y) = 0$, hence $\Delta > 0$, therefore it is a maximum point.

3c-i) $M(x,y) = xy$ & $N(x,y) = x^2 + y$, we have $\frac{\partial M}{\partial y} = x$, $\frac{\partial N}{\partial x} = 2x$,it is clear that integrating factor is $\mu = y$ such that $\frac{d\mu}{\mu} = \int \frac{N_x - M_y}{M} dy = \int \frac{1}{y} dy = \ln y$. Thus the new equation $(y^2x)dx + (x^2y + y^2)\,dy = 0$ is exact, therefore $f(x,y) = \frac{y^2x^2}{2} + \frac{y^3}{3}$.

ii) $u(x) = \exp(\int -x\,dx) = \exp(-x^2/2)$, $\int u(x)\,q(x)\,dx = \int (\cos x)dx$ $= \sin x$, therefore the solution is $y\exp(-x^2/2) = \sin x + c$

iii) The D.E is homogeneous, thus put $y = vx$, so $dy = vdx + xdv$, from which the D.E. will be $\frac{vdx+xdv}{dx} = \frac{(vx)^2}{vx^2 - x^2} = \frac{v^2}{v-1}$, therefore $-v\,dx + (v-1)x\,dv = 0$, thus $\ln x + \ln(y/x) - y/x = c$ is the solution.

iv) The characteristic equation is $r^3 - 4r^2 + 5r - 2 = 0$, therefore $r = 2, 1, 1$thus $y_c = (c_1 e^{2x} + c_2 e^{x} + c_2 xe^{x})$

4a-i) Put x = r cosθ, y = r sinθ, therefore

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\ dy\,dx = \int_0^3\int_0^{\pi} r^2\ dr\,d\theta = 9\pi$$

ii) Since the contour integration is the line joining (0,2) and (1,4) such that y = 2x + 2, therefore dy = 2dx, hence

$$\int_c \sin(\pi y)\,dx + yx^2 dy = \int_0^1 \sin(2\pi x)\,dx + (2x+2)x^2(2)dx$$

$$= \int_0^1 \sin(2\pi x)\,dx + 4(x^3+x^2)dx = \frac{-\cos(2\pi x)}{2\pi} + (x^4+\frac{4}{3}x^3)\Big|_0^1 = \frac{7}{3}$$

4b-i) The characteristic equation is $r^2-3r+2=0$, therefore r=2, 1, thus $y_c = (c_1 e^{2x} + c_2 e^x)$, and $y_1(x) = e^{2x}$, $y_2(x) = e^x$, therefore $W(y_1, y_2) = y_2' y_1 - y_2 y_1' = -e^{3x}$, $g(x) = \frac{2}{1+e^{-x}}$,

$\int \frac{y_2 g(x)}{W(y_1,y_2)}dx = -2\int \frac{e^{-2x}}{1+e^{-x}}dx = 2[e^{-x} - \ln(1+e^{-x})]$, $\int \frac{y_1 g(x)}{W(y_1,y_2)}dx$

$= -2\int \frac{e^{-x}}{1+e^{-x}}dx = 2\ln(1+e^{-x})$, but $Y_P(x) = -y_1\int \frac{y_2 g(x)}{W(y_1,y_2)}dx + y_2\int \frac{y_1 g(x)}{W(y_1,y_2)}dx$, thus $Y = Y_c + Y_P$

ii) The characteristic equation is $r^2 - 2r + 1 = 0 \Rightarrow r$ = -1, -1, so

$y_c = (c_1 e^{-x} + c_2 xe^{-x})$, $Y_P(x) = \frac{1}{D^2 - 2D + 1}[(x^2-1)e^{2x} + (3x+4)e^x]$

$= e^{2x}\frac{1}{D^2 + 2D + 1}(x^2 - 1) - e^x\frac{1}{D^2}(3x+4) = e^{2x}[1-2D + 3D^2](x^2 - 1)$

$- e^x (x^3/2 + 2x^2)$

iii) The characteristic equation is $r^2 + 5r + 4 = 0 \Rightarrow$ r = -4, -1,

therefore $y_c = (c_1 e^{-4x} + c_2 e^{-x})$, $Y_P = \frac{1}{D^2 + 5D + 4} e^{5x} \cos 2x$

$= e^{5x} \frac{1}{(D+5)^2 + 5(D+5) + 4} \cos 2x = e^{5x} \frac{1}{D^2 + 15D + 54} \cos 2x$

$= e^{5x} \frac{1}{15D + 50} \cos 2x = e^{5x} \frac{(15D - 50)}{225D^2 + 2500} \cos 2x$

$= e^{5x} \frac{(-30 \sin 2x - 50 \cos 2x)}{225(-4) + 2500}$

Benha University
Faculty of Engineering- Shoubra
Mechanical Engineering تخلفات
1st Year Mechanical (production)

Final Term Exam
Date:2nd of January 2012
Mathematics 1
Duration : 3 hours

1-a) Test the following series for convergence:

i) $\sum_{n=1}^{\infty}[\frac{5n^2-3n^3}{7n^3+2}]^{5n}$ ii) $\sum_{n=1}^{\infty}\frac{n^2}{(3n+1)!}$ iii) $\sum_{n=1}^{\infty}\frac{7}{3n^2+2n}$

(12marks)

1-b) Solve the following differential equations:

i) $(y + \ln(x))dx + (x+y^2)\, dy = 0$ ii) $y` = (y/2x) - (xy)^3$

iii) $y`` + y = 1 + \tan x$ iv) $y`` + 2y` + 2y = e^x \sin^2(2x)$

v) $y` = (x/y) + \tan(x/y)$ vi) $\frac{dy}{dx} = \frac{x+y+3}{x-y+1}$

(24 marks)

2-a) Expand the function $f(x, y) = \sin^{-1}(\frac{x+y}{x-y})$ using Taylor series about (0,1)

(8marks)

2-b) Find the interval of convergence for the following series:

i) $\sum_{n=1}^{\infty}\frac{3^n}{(n^2+1)(x-2)^n}$ ii) $\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!\,x^{2n+1}}$ iii) $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^3+1}$

(12 marks)

2-c) Find Envelope of $f(x, y, t) = \frac{x}{t} + \frac{y}{1-t} = 1$, t is the parameter

(6 marks)

2-d) Find the dimensions of the rectangular box with the largest volume if the total surface area is 64 cm^2.

(6 marks)

3-a) Find the first four terms in each portion of the series solution around x = 0 for the following differential equation $(1+x^2)\, y`` - xy` + 6y = 0$

(10 marks)

3-b) Find all relative extrema and saddle points for $f(x, y) = 3x^2 + y^2 + 9x - 4y + 6$

(8 marks)

4-a) Evaluate the following integrals

i) $\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \sqrt{x^2 + y^2}\, dy\, dx$ ii) $\int_c \sin(\pi y)\, dx + yx^2 dy$, c is line segment from (0,2) to (1,4)

(12 marks)

4-b) Find the volume of the parallelepiped spanned by the vectors

u = (1,0,2) v = (0,2,3) w = (0,1,3)

(7 marks)

Model answer

1a-

i) $\lim_{n\to\infty}\sqrt[n]{[\frac{5n^2-3n^3}{7n^3+2}]^{5n}} = \lim_{n\to\infty}[\frac{5n^2-3n^3}{7n^3+2}]^5 = [-\frac{3}{7}]^5 < 1 \Rightarrow \sum_{n=1}^{\infty}[\frac{5n^2-3n^3}{7n^3+2}]^{5n}$

is convergent.

1a-ii) Since $U_{n+1} = \frac{(n+1)^2}{(3n+4)!}$, hence $\frac{U_{n+1}}{U_n} = \frac{(n+1)^2}{(3n+4)!}\frac{(3n+1)!}{n^2}$

$= \frac{(n+1)^2}{(3n+4)(3n+3)(3n+2)n^2} \Rightarrow \lim_{n\to\infty}\frac{U_{n+1}}{U_n} = 0 < 1$, so $\sum_{n=1}^{\infty}\frac{n^2}{(3n+1)!}$ is convergent,

1a-iii) The series $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty}\frac{7}{3n^2}$ is convergent, therefore $\sum_{n=1}^{\infty}\frac{7}{3n^2+2n}$ is convergent as $\frac{7}{3n^2} > \frac{7}{3n^2+2n}$.

1b-i) $(y + \ln(x))dx + (x+y^2)\,dy = 0$ is exact D.E. since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$,

thus $\frac{\partial f}{\partial x} = M(x,y) = (y + \ln(x)) \Rightarrow f(x,y) = xy + x\mathrm{Ln}\,x - x + \phi(y)$, thus

$\frac{\partial f}{\partial y} = x + \phi'(y) = x+y^2$, hence $\phi(y) = y`` + 2y` + 2y = e^x \sin^2(2x)y^3/3$,

therefore solution is $f(x,y) = xy + x\mathrm{Ln}\,x - x + y^3/3 + c$.

1b-ii) $y' = (y/2x) - (xy)^3$ is Bernoulli D.E. , therefore $y^{-3} y' - y^{-2}/2x = -x^3$. Put $z = y^{-2}$, therefore $z' = -2 y^{-3} y'$, thus $z' + z/x = 2x^3$ which is linear D.E. whose solution is $zx = -2 x^5/ 5+c$, hence $xy^{-2} = -2 x^5/ 5 + c$ is the solution of D.E.

1b-iii) $y'' + y = 1+ \tan x$ has homogeneous and particular solution so that the characteristic equation is $m^2 +1=0 \Rightarrow m = -i, i$, thus $y_H = (c_1 \cos x + c_2 \sin x)$ and so the particular solution is $y_P = u_1(x) \cos x + u_2(x)\sin x$, $y_1(x) = \cos x$, $y_2(x) = \sin x$ where $u_1(x) = -\int \frac{y_2 g(x)}{W(y_1, y_2)} dx$,

$u_2(x) = \int \frac{y_1 g(x)}{W(y_1, y_2)} dx$, $W(y_1, y_2) = y_2' y_1 - y_2 y_1' = 1$, $g(x) = 1+ \tan x$,

therefore $u_1(x) = -\int \frac{\sin x\,(1+\tan x)}{1} dx = -\cos x + \ln(\sec x + \tan x) - \sin x$,

$u_2(x) = \int \frac{\cos x\,(1+\tan x)}{1} dx = \sin x - \cos x$

1b-iv) $y'' + 2y' + 2y = e^x \sin^2(2x)$ has homogeneous and particular solution so that the characteristic equation is $m^2 + 2m+2=0 \Rightarrow m = -1 \pm i$, thus $y_H = e^{-x} (c_1 \cos x + c_2 \sin x)$and so the particular solution is $y_P = \frac{1}{D^2+2D+2} e^x \sin^2 x = \frac{1}{D^2+2D+2} (\frac{e^x}{2})(1-\cos 2x) = \frac{e^x}{2}[\frac{1}{5} - \frac{(8\sin 2x + \cos 2x)}{65}]$

1b-vi) Since $\frac{dy}{dx} = \frac{x+y+3}{x-y+1}$ is non homogeneous equation. To solve this differential equation, we have to follow these steps

(1) We have to get the point of intersection between x + y + 3 = 0, x- y + 1 = 0 which is (-2,-1),

(2) Put x = X-2, y = Y-1, dx=dX,dy = dY in the above differential equation, then $\frac{dY}{dX} = \frac{X+Y}{X-Y}$, so it is a homogeneous equation,

(3) Put Y = vX, & dY = vdX + Xdv, so $\frac{vdX+Xdv}{dX} = \frac{X+vX}{X-vX} = \frac{1+v}{1-v}$

(4) Integrate $\frac{dX}{X} = \frac{(1-v)dv}{1+v^2}$, then put X=x+2, $v = \frac{Y}{X} = \frac{y+1}{x+2}$ so that the solution of the differential equation is Ln(x+2) = $\tan^{-1}(\frac{y+1}{x+2})$ - $\frac{1}{2}\ln(\frac{(y+1)^2+(x+2)^2}{(x+2)^2})$ +C

2a) Since f(x, y) = $\tan^{-1}(\frac{x+y}{x-y})$, therefore $f_x = \frac{-y}{x^2+y^2}$, $f_y = \frac{x}{x^2+y^2}$,

$f_{xx} = \frac{2xy}{(x^2+y^2)^2}$, $f_{xx} = \frac{2xy}{(x^2+y^2)^2}$, $f_{yy} = \frac{-2xy}{(x^2+y^2)^2}$, and $f_{xy} = \frac{y^2-x^2}{(x^2+y^2)^2}$.

At (0,1) , f(0,1) = $-\frac{\pi}{4}$, f_x = -1, f_y = 0, $f_{xx} = f_{yy} = \frac{2xy}{(x^2+y^2)^2}$ = 0, f_{xy} = 1,

then by substituting in Taylor formula, we get f(x, y) = $-\frac{\pi}{4}$ -x + x(y-1)

2b-i) Since $U_n = \frac{3^n}{(n^2+1)(x-2)^n}$, and $U_{n+1} = \frac{3^{n+1}}{((n+1)^2+1)(x-2)^{n+1}}$,

hence $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{3^{n+1}(n^2+1)(x-2)^n}{3^n((n+1)^2+1)(x-2)^{n+1}}\right| = \left|\frac{3(n^2+1)}{((n+1)^2+1)(x-2)}\right|$,therefore

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{3(n^2+1)}{((n+1)^2+1)(x-2)}\right| = \lim_{n\to\infty}\left|\frac{3}{(x-2)}\right| < 1$$ to be convergent,

hence $|x-2| > 3$, thus x > 5 or x < -1 is the interval of convergence.

2b-ii) Since $U_n = \frac{(-1)^n}{(2n+1)!\,x^{2n+1}}$, and $U_{n+1} = \frac{(-1)^{n+1}}{(2n+3)!\,x^{2n+3}}$, hence

$$\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{(2n+1)!\,x^{2n+1}}{(2n+3)!\,x^{2n+3}}\right| = \left|\frac{1}{(2n+3)(2n+2)x^2}\right| \Rightarrow \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = 0$$ is

convergent for all x.

2b-iii) Since $U_n = \frac{(x-2)^n}{n^3+1}$, and $U_{n+1} = \frac{(x-2)^{n+1}}{(n+1)^3+1}$, hence the $\left|\frac{U_{n+1}}{U_n}\right| =$

$$\left|\frac{[(n)^3+1](x-2)^{n+1}}{(x-2)^n[(n+1)^3+1]}\right| = \left|\frac{[(n)^3+1]}{[(n+1)^3+1](x-2)}\right| \Rightarrow \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| < 1,$$ thus

$|x-2| > 1$, so that x > 3, x < 1.

2c) Differentiate w.r.t. t such that $-x/t^2 - y(1-t)^{-2}(-1) = 0$, therefore

$t = \frac{1}{1\pm\sqrt{\frac{y}{x}}}$ & $1-t = \frac{\pm\sqrt{\frac{y}{x}}}{1\pm\sqrt{\frac{y}{x}}}$, thus the envelope is $(1\pm\sqrt{\frac{y}{x}})(x\pm\sqrt{xy}) = 1$

2d) $f(x,y,z) = xyz$, $\phi(x,y,z) = 2[xy+yz+xz] = 64$ and $f_x = \lambda\phi_x$, $f_y = \lambda\phi_y$

and $f_z = \lambda\phi_z$, therefore, $x = y = z = \sqrt{\frac{32}{3}}$, so largest volume $= \sqrt{(\frac{32}{3})^3}$.

3a) Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ $y'(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$, $y''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$,

therefore $(1+x^2) \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} - x \sum_{n=1}^{\infty} n\, a_n x^{n-1} + 6 \sum_{n=0}^{\infty} a_n x^n = 0$

3b) $f_x = 6x + 9 = 0$, $f_y = 2y + 4 = 0$, therefore (-3/2 , 2) is the critical point, hence (0,0), and $f_{xx} = 6$, $f_{yy} = 2$, $f_{xy} = 0$. At (-3/2 , 2), $\Delta = 12 > 0$ (minimum point)

4a-i) Put $x = r\cos\theta$, $y = r\sin\theta$, therefore $\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \sqrt{x^2 + y^2}\, dy\, dx =$

$\int_0^3 \int_0^{\pi} r^2\, dr\, d\theta = 9\pi$

4a-ii) Since the contour integration is the line joining (0,2) and (1,4) such that y = 2x + 2, therefore dy = 2dx, hence $\int_c \sin(\pi y)\, dx + yx^2 dy$

$= \int_0^1 \sin(2\pi x)\, dx + (2x+2)x^2(2)dx = \int_0^1 \sin(2\pi x)\, dx + 4(x^3 + x^2)dx$

$= \left. \frac{-\cos(2\pi x)}{2\pi} + (x^4 + \frac{4}{3}x^3) \right|_0^1 = \frac{7}{3}$.

4b) Volume = $\bar{u} \bullet (\bar{v} \times \bar{w}) = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 1 & 3 \end{vmatrix} = 3$.

Question Bank

1- Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty}\frac{(-10)^n}{4^{2n+1}(n+1)},\quad \sum_{n=1}^{\infty}\frac{n^2}{(2n-1)!},\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+1)},\quad \sum_{n=1}^{\infty}\frac{n+2}{2n+7},$$

$$\sum_{n=1}^{\infty}\frac{n^n}{3^{(1+2n)}},\quad \sum_{n=1}^{\infty}[\frac{5n-3n^3}{7n^3+2}]^n,\quad \sum_{n=1}^{\infty}\frac{1}{3^n+n},\quad \sum_{n=1}^{\infty}\frac{n}{n^2-\cos^2(n)},$$

$$\sum_{n=1}^{\infty}\frac{n^2+2}{n^4+5},\quad \sum_{n=1}^{\infty}\frac{4n^2+n}{\sqrt[3]{n^7+n^3}},\quad \sum_{n=0}^{\infty}\frac{(-1)^{n-3}\sqrt{n}}{n+4},\quad \sum_{n=1}^{\infty}\frac{\cos(n\pi)}{\sqrt{n}},$$

$$\sum_{n=1}^{\infty}[\frac{2n^2+1}{n^2+1}]^n,\quad \sum_{n=1}^{\infty}\frac{7}{3n^2+2n},\quad \sum_{n=2}^{\infty}\frac{1}{n\,Ln(n)},\quad \sum_{n=1}^{\infty}n\,e^{n^2}$$

$$\sum_{n=1}^{\infty}\frac{5^n+7^n}{3^n+2^n},\quad \sum_{n=2}^{\infty}\frac{Ln(n)}{n},\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{3^n(2n!)^2},\quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,2^n},$$

2- Find interval of convergence for the following series and determine the behavior of the series at the endpoints of the interval. State clearly where the series converges absolutely, where it converges conditionally, and where it diverges

$$.\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2},\quad \sum_{n=1}^{\infty}\frac{x^n}{n!},\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!\,x^{2n+1}},\quad \sum_{n=1}^{\infty}n^3x^{2n},$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{3^n(n!)^2},\quad \sum_{n=1}^{\infty}\frac{(-1)^n(x-6)^n}{n\,3^n},\quad \sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^{2n}}{2n!},\quad \sum_{n=1}^{\infty}\frac{(x-3)^n}{n!}$$

3) Find the critical points and classify the following functions

a) $f(x, y) = x^3 + x y^2 - 3x^2 - 4 y^2 + 4$

b) $f(x, y) = x^2 + y^2 + x^2 y + 4$

c) $f(x, y) = 9 - 2x + 4y - x^2 - 4 y^2$

d) $f(x, y) = xy - 2x - y$

e) $f(x, y) = 1 + xy - x - y$

f) $f(x, y) = \dfrac{x^2 y^2 - 8x + y}{xy}$

4) A box having a square base and an open top is to contain 108 cubic feet. What should its dimensions be so that the material to make it will be a minimum? That is, what dimensions will cost the least?

5) Find the dimensions of the rectangle that, for a given perimeter, will have the largest area.

6) Find the dimensions of the rectangle with the most area that can be inscribed in a semi-circle of radius r. Show, in fact, that the area of that rectangle is r^2

7) Find the maximum and minimum values of f subject to the given constraints

a) $f(x, y) = x^2 - y^2$, $x^2 + y^2 = 1$

b) $f(x, y) = x^2y$, $x^2 + 2y^2 = 6$

c) $f(x, y) = 2x + 6y + 10z$, $x^2 + y^2 + z^2 = 35$

8) Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

9) Find the dimensions of the rectangular box with the largest volume if the total surface area is 64 cm^2.

10) A cardboard box without a lid is to have volume of 32,000 cm^3, find the dimensions that minimize the amount of cardboard used.

11) Solve the following Differential equations

$y` = \frac{-2x+5y}{2x+y}$, $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$,

$y` = y + \sin(x)$, $xy` + y = -x^3$, $y` + y/x = -2$,

$y` = e^{x-y}$, $xy` - \sin(x)/y = y\,Ln(x)$, $y` = \dfrac{\cos y - ye^x}{e^x + x\sin y}$,

$y` = x^3 - (4/x)y$, $y` = \dfrac{xy + x^2}{y^2}$, $y` = (x/y) + \tan(x/y)$,

$y` = x(2Lnx+1)/(\sin y + y\cos y)$, $y - xy` = a\,(y^2 + y`)$,

$y` = y\,e^x - 2e^x + y - 2$, $y` = x(\cos x)\,y$, $y` = (y/2x) - (xy)^3$,

$y` = -2xy + 2x$, $y` = y\tan x - \cos x$, $y^2 - xy + x^2 y` = 0$,

$y` = \dfrac{xy}{1+x^2} + \sqrt{\dfrac{1+x^2}{1-x^2}}$, $\dfrac{x}{\sqrt{y^2+x^2}} + \dfrac{yy'}{\sqrt{y^2+x^2}} = 0$, $\dfrac{1+y}{1+x} + y` = 0$,

$y` = \sqrt{y-x}$ (Hint: put $\sqrt{y-x} = t$), $x^2 + y/x + Ln(xy)\,y` = 0$,

$1+y + (2y + 2y^2)y` = 0$, $1 - \dfrac{x}{x^2+y^2} - \dfrac{yy'}{x^2+y^2} = 0$

$y`` + 2y` + 2y = e^x \sin^2(2x)$, $y`` + y = \sec(x)$,

$y``` + y`` - y` - y = e^x$, $y`` + y = 1 + \tan x$, $y```` + y = 0$

$16y`` + 8y` + y = 0$,   $y`` + 5y` + 6y = 2 - x + 3x^2$

$y`` + 3y` + 2y = e^{2x}\cos x$,   $y`` + 3y` + 2y = \dfrac{1}{(1+e^{x})^{2}}$

12) Solve the differential equation (the so called driven and damped harmonic oscillator) $y'' + \gamma\, y' + \omega^2 y = h\cos(\Omega x)$, where $\gamma > 0$, ω, h, and Ω are constants with some given numerical values. What happens in the limit $\gamma \to 0$ if $\omega = \Omega$?

13) Determine the general solution of the following systems of linear differential equations

$y` = -6y + 4z$,   $z` = -8y + 2z$

$y` = -8y - 10z$,   $z` = 5y + 7z$, $y(0) = 1$, $z(0) = 0$

$y` = (\sqrt{2} - 1)\,y + (2/\sqrt{3} + 1)z$,   $z` = -\sqrt{3}\; y + (\sqrt{2} + 1)\,z$

14) Find Series solution for the following D.E.`s about x = 0

$2x^2\, y`` - x\, y` + (1 + x^2)\, y = 0$,   $9x^2\, y`` - (4+x)\, y = 0$

$25\,x^2\,y`` + 25\,x\,y` - (x+1)y = 0$,      $xy` - y` + xy = 0$

$x^2\,y`` - x\,y` + (x+1)y = 0$,      $xy`` - y = 0$, $y`` - xy = 20$

15) Given R = (x, y, z) so that r = |R| = $\sqrt{x^2+y^2+z^2}$. Show that $\nabla(r^n) = n\,r^{n-2}\,R$, for any integer n, then deduce grad (r), grad (r^2), grad (1/r),grad (c.R), curl ($c\times R$) where c is a constant vector. Find the values of n for which $\nabla^2(r^n) = 0$, div(r^n (c $\times$ R)); (ii) curl(r^n (c $\times$ R)).

16) Prove the identities

(i) curl grad f = 0; (ii) curl(f F) = f curl F + gradf $\times$ F; (iii) div(f F) = f divF + (grad f) . F, Let f and g be smooth scalar fields and F and G smooth vector fields.

17) Let F = (x^2y, yz, x + z). Find curl curl F; grad div F.

18) Let F = (sin(y - z), x cos(y - z) – 1, - x cos(y - z) + z). Show that curl F = 0 and find a scalar field ϕ such that F = grad ϕ.

Important Rules

Important rules on trigonometric functions

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sin 2x = 2 \sin x \cos x$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cos 2x = \cos^2 x - \sin^2 x$
$\cos x \cos y = [\cos(x-y) + \cos(x+y)]/2$	$\cos^2 x + \sin^2 x = 1$
$\sin x \sin y = [\cos(x-y) - \cos(x+y)]/2$	$\cos^2 x = \frac{(1+\cos 2x)}{2}$
$\sin x \cos y = [\sin(x-y) + \sin(x+y)]/2$	$\sin^2 x = \frac{(1-\cos 2x)}{2}$
$\sin x = \cos(90-\theta)$	$\tan x = \cot(90-\theta)$
$\cos x = \sin(90-\theta)$	$\cot x = \tan(90-\theta)$
$\tan^2 x = \sec^2 x - 1$	$\cot^2 x = \csc^2 x - 1$

Important rules on hyperbolic functions

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, $\cosh x + \sinh x = e^x$
$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, $\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$,
$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
$\sinh 2x = 2 \sinh x \cosh x$, $\cosh 2x = \cosh^2 x + \sinh^2 x$
$\cosh^2 x = \frac{(1+\cosh 2x)}{2}$, $\sinh^2 x = \frac{(\cosh 2x-1)}{2}$,
$\cosh^2 x - \sinh^2 x = 1$, $\tanh^2 x = 1-\mathrm{sech}^2 x$, $\coth^2 x = 1+\mathrm{csch}^2 x$

Important rules on logarithmic functions

1) $\log_a[xy] = \log_a[x] + \log_a[y]$
2) $\log_a[x/y] = \log_a[x] - \log_a[y]$
3) $\log_a[x^y] = y\log_a[x]$
4) $\log_a[x] = \ln x / \ln a$

Important rules on derivatives

Function	First derivative
$c^{f(x)}$	$\frac{d}{dx}[c^{f(x)}] = \frac{df}{dx} c^{f(x)} \ln c$
$\ln(f(x))$	$\frac{d}{dx}[\ln(f(x)] = \frac{df}{dx} / f(x)$
$\sin f(x)$	$\frac{d}{dx}[\sin f(x)] = \frac{df}{dx}[\cos f(x)]$
$\cos f(x)$	$\frac{d}{dx}[\cos f(x)] = -\frac{df}{dx}[\sin f(x)]$

tan f(x)	$\frac{d}{dx}[\tan f(x)] = \frac{df}{dx} \sec^2 f(x)$
sec f(x)	$\frac{d}{dx}[\sec f(x)] = \frac{df}{dx} \sec f(x) \tan f(x)$
cosec f(x)	$\frac{d}{dx}[\text{cosec } f(x)] = -\frac{df}{dx} \text{cosec } f(x) \cot f(x)$
cot f(x)	$\frac{d}{dx}[\cot f(x)] = -\frac{df}{dx} \text{cosec}^2 f(x)$
arccosf(x)	$\frac{d}{dx}[\arccos f(x)] = -\frac{1}{\sqrt{1-[f(x)]^2}} \frac{df}{dx}$
arcsin f(x)	$\frac{d}{dx}[\arcsin f(x)] = \frac{1}{\sqrt{1-[f(x)]^2}} \frac{df}{dx}$
arctan f(x)	$\frac{d}{dx}[\arctan f(x)] = \frac{1}{1+[f(x)]^2} \frac{df}{dx}$
arccot f(x)	$\frac{d}{dx}[\text{arccot } f(x)] = -\frac{1}{1+[f(x)]^2} \frac{df}{dx}$
arcsec f(x)	$\frac{d}{dx}[\text{arcsec } f(x)] = \frac{1}{[f(x)]\sqrt{[f(x)]^2-1}} \frac{df}{dx}$
arccsc f(x)	$\frac{d}{dx}[\text{arccosec } f(x)] = -\frac{1}{[f(x)]\sqrt{[f(x)]^2-1}} \frac{df}{dx}$

cosh f(x)	$\frac{d}{dx}[\cosh f(x)] = \frac{df}{dx}[\sinh f(x)]$
sinh f(x)	$\frac{d}{dx}[\sinh f(x)] = \frac{df}{dx}[\cosh f(x)]$
tanh f(x)	$\frac{d}{dx}[\tanh f(x)] = \frac{df}{dx}\ \mathrm{sech}^2 f(x)$
sech f(x)	$\frac{d}{dx}[\mathrm{sech}\ f(x)] = -\frac{df}{dx}\ \mathrm{sech}\ f(x)\ \tanh f(x)$
csch f(x)	$\frac{d}{dx}[\mathrm{csch}\ f(x)] = -\frac{df}{dx}\ \mathrm{csch}\ f(x)\ \coth f(x)$
coth f(x)	$\frac{d}{dx}[\coth f(x)] = -\frac{df}{dx}\ \mathrm{cosech}^2 f(x)$
$\cosh^{-1} f(x)$	$\frac{d}{dx}[\cosh^{-1} f(x)] = \frac{1}{\sqrt{[f(x)]^2 - 1}} \frac{df}{dx}$
$\sinh^{-1} f(x)$	$\frac{d}{dx}[\sinh^{-1} f(x)] = \frac{1}{\sqrt{1 + [f(x)]^2}} \frac{df}{dx}$
$\tanh^{-1} f(x)$	$\frac{d}{dx}[\tanh^{-1} f(x)] = \frac{1}{1 - [f(x)]^2} \frac{df}{dx}$
$\coth^{-1} f(x)$	$\frac{d}{dx}[\coth^{-1} f(x)] = \frac{1}{1 - [f(x)]^2} \frac{df}{dx}$

$\text{sech}^{-1} f(x)$	$\frac{d}{dx}[\text{sech}^{-1} f(x)] = -\frac{1}{f(x)\sqrt{1-[f(x)]^2}}\frac{df}{dx}$
$\text{csch}^{-1} f(x)$	$\frac{d}{dx}[\text{csch}^{-1} f(x)] = -\frac{1}{f(x)\sqrt{1+[f(x)]^2}}\frac{df}{dx}$

Table of important integrals

$\int u^n \, du = \frac{1}{n+1}u^{n+1} + C$
$\int u^{-1} \, du = \int \frac{1}{u} \, du = \ln\lvert u\rvert + C$
$\int e^u \, du = e^u + C$
$\int a^u \, du = \frac{1}{\ln a}a^u + C$
$\int \sin u \, du = -\cos u + C$
$\int \cos u \, du = \sin u + C$
$\int \sec^2 u \, du = \tan u + C$

$\int \sec u \tan u \, du = \sec u + C$
$\int \csc^2 u \, du = -\cot u + C$
$\int \csc u \cot u \, du = -\csc u + C$
$\int \tan u \, du = \ln\lvert\sec u\rvert + C = -\ln\lvert\cos u\rvert + C$
$\int \sec u \, du = \ln\lvert\sec u + \tan u\rvert + C$
$\int \frac{1}{\sqrt{a^2 - u^2}} \, du = \sin^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{1}{a^2 + u^2} \, du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{1}{u\sqrt{u^2 - a^2}} \, du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{1}{a^2 - u^2} \, du = \frac{1}{2a}\ln\left\lvert\frac{a+u}{a-u}\right\rvert + C$
$\int \sin^{-1} u \, du = u\sin^{-1} u + \sqrt{1 - u^2} + C$

$\int \cos^{-1} u \, du = u \cos^{-1} u - \sqrt{1-u^2} + C$
$\int \tan^{-1} u \, du = u \tan^{-1} u - \frac{1}{2} \ln\left(u^2+1\right) + C$
$\int \cot^{-1} u \, du = u \cot^{-1} u + \frac{1}{2} \ln\left(u^2+1\right) + C$
$\int \sec^{-1} u \, du = u \sec^{-1} u - \ln\left(u + \sqrt{u^2-1}\right) + C$
$\int \csc^{-1} u \, du = u \csc^{-1} u + \ln\left(u + \sqrt{u^2-1}\right) + C$
$\int e^{au} \, du = \frac{1}{a} e^{au} + C$
$\int u e^{au} \, du = \frac{1}{a^2} (au-1) e^{au} + C$
$\int u^2 e^{au} \, du = \frac{1}{a^3} \left(a^2u^2 - 2au + 2\right) e^{au} + C$
$\int u^2 e^{au} \, du = \frac{1}{a^3} \left(a^2u^2 - 2au + 2\right) e^{au} + C$
$\int e^{au} \sin bu \, du = \frac{1}{a^2+b^2} e^{au} (a \sin bu - b \cos bu) + C$

$\int e^{au}\cos bu\,du = \frac{1}{a^2+b^2}e^{au}(a\cos bu + b\sin bu) + C$
$\int \ln u\,du = u\ln u - u + C$
$\int af(u)\,du = a\int f(u)\,du$
$\int [f(u)+g(u)]\,du = \int f(u)\,du + \int g(u)\,du$
$\int [f(u)-g(u)]\,du = \int f(u)\,du - \int g(u)\,du$
$\int [af(u)+bg(u)]\,du = a\int f(u)\,du + b\int g(u)\,du$
$\int u\,dv = uv - \int v\,du$ *(Integration by parts)*
$\int_a^b f(u)\,du = -\int_b^a f(u)\,du$ *(Definite integral)*
$\int_a^c f(u)\,du = \int_a^b f(u)\,du + \int_b^c f(u)\,du$ *(Definite integral)*